

ON THE FIBER PRODUCT OF RIEMANN SURFACES II

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ABSTRACT. Given non-constant holomorphic maps $\beta_j : S_j \rightarrow S_0$, $j = 1, 2$, between closed Riemann surfaces, there is associated its fiber product (in the set theoretical sense), which may or not be connected and when it is connected it may or not be irreducible. A Fuchsian group description of the irreducible components of the fiber product is given and, as a consequence, we show that if one of the maps β_j is regular, then all irreducible components are isomorphic. In the case that the fiber product is connected (for instance, if S_0 has genus zero by results of Fulton-Hansen), we provide sufficient conditions for it to be irreducible; examples are provided to see that these conditions are not necessary in general. We define the (strong) field of moduli of the fiber product and see that it coincides with the minimal field containing the fields of moduli of the starting pairs (S_1, β_1) and (S_2, β_2) . We also study an isogenous decomposition of the Jacobian variety of the fiber product in some cases.

1. INTRODUCTION

In the category of sets there is a construction called the *fiber product* which satisfies certain universality property, but such a construction may not always be realizable in some subcategories. It is known that the fiber product of algebraic varieties (or schemes) is again an algebraic variety (or scheme) [17] and this object has been a main tool for constructing examples and counterexamples in algebraic geometry.

In this paper we are mainly interested in the case of closed Riemann surfaces. Let us consider the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ (see Section 2.2) of the two pairs (S_1, β_1) and (S_2, β_2) , where S_0 , S_1 and S_2 are closed Riemann surfaces and $\beta_1 : S_1 \rightarrow S_0$ and $\beta_2 : S_2 \rightarrow S_0$ are non-constant holomorphic maps. In general, $S_1 \times_{(\beta_1, \beta_2)} S_2$ is not a closed Riemann surface, but a *singular Riemann surface* (see Section 2.1 for the precise definition). By deleting all singular points of the fiber product (this is a finite set), we obtain a finite collection of analytically finite Riemann surfaces T_1, \dots, T_n . Each T_j can be compactified (by adding its punctures) to obtain a unique, up to biholomorphism, closed Riemann surface, called an *irreducible component* of the fiber product. In [11] the first author observed that any two irreducible components of lowest genus (if different from one) define isomorphic closed Riemann surfaces (for genus one they are isogenous elliptic curves). The aim of this paper is to continue with the study of such fiber products.

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The paper is organized as follows. In Section 2.1 we provide the definition of singular Riemann surfaces. In Section 2.2 we recall the definition of the fiber product; in the case of Riemann surfaces we prove that it is a singular Riemann surface (Proposition 2.1). In Section 2.3 we provide a Fuchsian group description of the analytically finite Riemann surfaces T_j obtained in the fiber product (Theorem 2.3) and, as a consequence, we observe that if one of the maps β_j is a regular branched covering, then the irreducible components are pairwise isomorphic Riemann surfaces (Corollary 2.5). In Section 2.4 we study conditions for the fiber product to be irreducible; so it has a structure of a closed Riemann surface. Fulton-Hansen in [9] have proved that, when S_0 has genus zero, the fiber product is connected. We provide some examples, when S_0 has positive genus, of non-connected fiber products. In the case that the fiber product is connected, we provide sufficient conditions for it to be irreducible (Theorem 2.6). Unfortunately, such sufficient conditions are not necessary, as we see from explicit examples. We also provide an upper bound for the number of irreducible components of the fiber product. In Section 3 we define an algebraic invariant of the fiber product, called the *strong field of moduli*. We observe that this field is the smallest field containing the fields of moduli of the two corresponding pairs (S_1, β_1) and (S_2, β_2) (Theorem 3.2). Some applications to dessins d'enfants are also discussed in Section 3.3. In Section 4, we study the Jacobian variety of the components of fiber products, when these are non-singular in the case that both maps β_j are regular. If, in addition, the maps β_j cover the Riemann sphere, we provide an isogenous decomposition of the Jacobian variety of the fiber product (Theorem 4.2). Finally, in Section 5 we provide a list of examples of fiber products. For many of the examples we have used MAGMA [2].

The results of this article are mostly based on the third author's Ph.D. thesis at the Universidad de Chile [28].

2. THE FIBER PRODUCT OF RIEMANN SURFACES

2.1. Singular Riemann surfaces. Let R_1, \dots, R_n be closed Riemann surfaces and C_1, \dots, C_m be pairwise disjoint finite subsets of $R_1 \cup \dots \cup R_n$. The space obtained by identify all points belonging to the same set C_j is called a *singular Riemann surface*; its *irreducible components* are the started Riemann surfaces R_j and its *singularities* are the m equivalence classes C_1, \dots, C_m . In the case that $n \geq 2$, one says that the singular Riemann surface is *reducible*; otherwise, it is called *irreducible*.

Observe that a singular Riemann surface has structure of compact topological space so that the complement of the singular points has structure of analytically finite Riemann surface. Moreover, the singular points have neighborhoods homeomorphic to a finite union of cones with a common vertex.

2.2. Fiber product of Riemann surfaces. Let us fix three closed Riemann surfaces S_1, S_2 and S_0 and two non-constant holomorphic maps $\beta_1 : S_1 \rightarrow S_0$ and $\beta_2 : S_2 \rightarrow S_0$. The *fiber product* (in the set theoretical setting) of the two pairs (S_1, β_1) and (S_2, β_2) is defined by

$$S_1 \times_{(\beta_1, \beta_2)} S_2 = \{(z_1, z_2) \in S_1 \times S_2 : \beta_1(z_1) = \beta_2(z_2)\} \subset S_1 \times S_2.$$

The above fiber product is a closed subset of the compact complex surface $S_1 \times S_2$, so it is compact. Moreover, if we denote by $\pi_j : S_1 \times_{(\beta_1, \beta_2)} S_2 \rightarrow S_j$ the projection $\pi_j(z_1, z_2) = z_j$, then there is a natural function $\beta : S_1 \times_{(\beta_1, \beta_2)} S_2 \rightarrow S_0$ so that $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2$.

In general, the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ is not a closed Riemann surface, but a singular Riemann surface as shown by the next.

A point $(z_1, z_2) \in S_1 \times_{(\beta_1, \beta_2)} S_2$ will be called a singular point if both z_1 is a critical point of β_1 and z_2 is a critical point of β_2 .

Proposition 2.1. *The fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ is a singular Riemann surface.*

Proof. If $(z_1^0, z_2^0) \in S_1 \times_{(\beta_1, \beta_2)} S_2$, then we consider local coordinates $z_j : U_j \subset S_j \rightarrow \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with $z_j(z_j^0) = 0$ so that $\beta_j(z_j) = z_j^{n_j}$. A neighborhood of the point (z_1^0, z_2^0) can be identified with $U = \{(z_1, z_2) \in \mathbb{D}^2 : z_1^{n_1} = z_2^{n_2}\} \subset \mathbb{C}^2$, where (z_1^0, z_2^0) is identified with $(0, 0)$.

Let d be the maximum common divisor between n_1 and n_2 . Then $n_j = dm_j$ for a suitable integer m_j . Now, if ω is a d -th primitive root of the unity, then we may see that

$$U = \left\{ (z_1, z_2) \in \mathbb{D}^2 : \prod_{k=0}^{d-1} (z_1^{m_1} - \omega^k z_2^{m_2}) = 0 \right\}$$

which is homeomorphic to a collection of d cones with common vertex at $(0, 0)$.

The previous observation, in particular, asserts the following facts. The points for which $d \geq 2$ are isolated (so a finite set) and the fiber product locally looks like the union of d cones glued together at the common vertex. The set of points in the fiber product for which $d = 1$ has structure of Riemann surface which might be non-connected; however, each connected component is analytically finite (a compact Riemann surface from which a finite number of points have been deleted). All the ensures asserts that the fiber product is a singular Riemann surface, as desired. \square

The above asserts that the complement of the singular points of the fiber product is a finite collection of analytically finite Riemann surfaces, say T_1^*, \dots, T_n^* . Next, for each of these analytically finite Riemann surfaces T_j^* we proceed to obtain a new analytically finite Riemann surface $T_j \subset T_j^*$ (this in order to have a good Fuchsian groups description later).

Let us denote by B the finite subset of S_0 consisting of those of points which are a branch value of either β_1 or β_2 . Set

$$S_0^* = S_0 - B, \quad S_1^* = S_1 - \beta_1^{-1}(B), \quad S_2^* = S_2 - \beta_2^{-1}(B).$$

It can be seen that $(S_1 \times_{(\beta_1, \beta_2)} S_2)^* = (S_1 \times_{(\beta_1, \beta_2)} S_2) - \beta^{-1}(B)$ is the fiber product of the two pairs (S_1^*, β_1) and (S_2^*, β_2) , where $\beta_1 : S_1^* \rightarrow S_0^*$ and $\beta_2 : S_2^* \rightarrow S_0^*$ are the natural restrictions. This fiber product turns out to be a Riemann surface (which might be disconnected) with a finite collection of connected components, each one an analytically finite Riemann surface, say T_1, \dots, T_n . It can be seen that each T_j corresponds to exactly one of the connected components T_j^* .

We have the natural restriction of $\pi_j : T_k \rightarrow S_j^*$ so that $\beta_1 \circ \pi_1 = \beta_2 \circ \pi_2 = \beta$. The following property is not hard to see (see the previous proof).

Remark 2.2. The universal property of the fiber product asserts the following. If X is an analytically finite Riemann surface and $p_j : X \rightarrow S_j^*$ are non-constant holomorphic maps so that $\beta_1 \circ p_1 = \beta_2 \circ p_2$, then there exists an irreducible component T_k of the fiber product (as in the above proof) and a holomorphic map $t : X \rightarrow T_k$ so that $p_j = \pi_j \circ T$. In fact, $t(x) = (p_1(x), p_2(x))$, for $x \in X$.

2.3. A Fuchsian group description of the fiber product. Let us keep with notation and definitions given above.

Let us assume S_0^* is a hyperbolic Riemann surface (the non-hyperbolic situation can be carried out in a similar way by replacing the hyperbolic plane by either the complex plane or the Riemann sphere). Let Γ_0 be a Fuchsian group, acting on the hyperbolic plane \mathbb{H}^2 , so that S_0^* is conformally equivalent to \mathbb{H}^2/Γ_0 .

As a consequence of basic covering theory, there is finite index subgroup Γ_j of Γ_0 so that the covering $\beta_j : S_j^* \rightarrow S_0^*$ is realized by any of the subgroups $\gamma\Gamma_j\gamma^{-1}$, where $\gamma \in \Gamma_0$.

Theorem 2.3. *Every component T_k is isomorphic to one of the quotients $\mathbb{H}^2/(\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1})$, where $\gamma \in \Gamma_0$.*

Proof. Let us fix subgroups $K_j = \gamma_j\Gamma_j\gamma_j^{-1}$, where $\gamma_j \in \Gamma_0$ for $j = 1, 2$. Set $K = K_1 \cap K_2$ and $R^* = \mathbb{H}^2/K$. We have covering maps $Q_j : R^* \rightarrow S_j^*$ induced by the inclusion of K inside K_j . Clearly, $\beta_1 \circ Q_1 = \beta_2 \circ Q_2$ and therefore, by the universal property of the fiber product, the surface \mathbb{H}^2/K is one of the components T_k . Conversely, each T_k is obtained in the above way by suitable choices of γ_1 and γ_2 . The result follows directly after noticing that for every $\gamma_1, \gamma_2 \in \Gamma_0$, the groups $\gamma_1\Gamma_1\gamma_1^{-1} \cap \gamma_2\Gamma_2\gamma_2^{-1}$ and $\Gamma_1 \cap (\gamma_1^{-1} \circ \gamma_2)\Gamma_2(\gamma_1^{-1} \circ \gamma_2)^{-1}$ provide isomorphic Riemann surfaces. \square

Corollary 2.4. *If one of the subgroups Γ_1 or Γ_2 is a normal subgroup of Γ , then all the components T_k are isomorphic Riemann surfaces.*

Proof. The normalizer N_j of Γ_j in Γ_0 (i.e., those $\gamma \in \Gamma_0$ so that $\Gamma_j = \gamma\Gamma_j\gamma^{-1}$) has finite index, say b , in Γ_0 . Then the theorem above asserts that in the collection T_1, \dots, T_n there are at most b non-isomorphic Riemann surfaces. In particular, if $b = 1$ then all of them are isomorphic. \square

If T is an analytically finite Riemann surface, then it is known that there is a unique closed Riemann surface \overline{T} containing it; see [20]. It follows that the irreducible component of the fiber product are given by $\overline{T}_1, \dots, \overline{T}_n$. In this way, the above can be stated as follows.

Corollary 2.5. *If one of the maps $\beta_j : S_j \rightarrow S_0$ is a regular branched cover, then all irreducible components of the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ are isomorphic.*

Proof. Let us assume that $\beta_2 : S_2 \rightarrow S_0$ is a regular branched cover. Then $\beta_2 : S_2^* \rightarrow S_0^*$ is a regular unbranched cover; so Γ_2 is a normal subgroup of Γ_0 . Thus $(\Gamma_1 \cap \gamma \Gamma_2 \gamma^{-1}) = \Gamma_1 \cap \Gamma_2$, for every $\gamma \in \Gamma_0$ and the result follows. \square

2.4. On the irreducibility of the fiber product. In [9] it was observed that the fiber product is connected when S_0 has genus zero. The following example shows that the fiber product might not be connected in the case that S_0 has positive genus.

Example 1 (Examples of non-connected fiber products).

- (1) *An example when S_0 has genus at least two.* Let us consider closed Riemann surfaces S_0 and S , both of genus at least two, and let $\pi : S \rightarrow S_0$ be an unbranched covering map of degree $d \geq 2$. Take $S_1 = S_2 = S$, $\beta_1 = \beta_2 = \pi$. If $S_1 \times_{(\beta_1, \beta_2)} S_2$ could be connected, then (as β has degree d^2) the genus of $S_1 \times_{(\beta_1, \beta_2)} S_2$ would be strictly bigger than of S . Now, by taking $P_j : S \rightarrow S_j$ equal to the identity we see that S covers $S_1 \times_{(\beta_1, \beta_2)} S_2$, providing a contradiction to the Riemann-Hurwitz formula (since the genus of S is strictly less than of $S_1 \times_{(\beta_1, \beta_2)} S_2$).
- (2) *An example when S_0 has genus one.* For $\lambda \in \mathbb{C} - \{0, 1\}$, we consider $S_1 = S_2 = S$, where

$$S = \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{P}_{\mathbb{C}}^3 : x_1^2 + x_2^2 + x_3^2 = 0, \lambda x_1^2 + x_2^2 + x_4^2 = 0\},$$

$$S_0 = \{[y_1 : y_2 : y_3 : y_4 : y_5] \in \mathbb{P}_{\mathbb{C}}^4 : y_3^2 = y_1 y_2, y_5(y_1 + y_2) + y_4^2 = 0, \lambda y_1 + y_2 + y_5 = 0\},$$

$$\beta_1 = \beta_2 = \pi, \text{ where } \pi([x_1 : x_2 : x_3 : x_4]) = [x_1^2 : x_2^2 : x_1 x_2 : x_3 x_4 : x_4^2].$$

Note that $\pi : S \rightarrow S_0$ is an unbranched two-fold cover whose deck group is cyclic generated by the involution $\tau([x_1 : x_2 : x_3 : x_4]) = [-x_1 : -x_2 : x_3 : x_4]$.

Then, the fiber product in this case is

$$\begin{aligned} X = \left\{ ([x_1 : x_2 : x_3 : x_4], [z_1 : z_2 : z_3 : z_4]) \in (\mathbb{P}_{\mathbb{C}}^3)^2 : \right. \\ \left. x_1^2 + x_2^2 + x_3^2 = 0, \lambda x_1^2 + x_2^2 + x_4^2 = 0, z_1^2 + z_2^2 + z_3^2 = 0, \lambda z_1^2 + z_2^2 + z_4^2 = 0, \right. \\ \left. [x_1^2 : x_2^2 : x_1 x_2 : x_3 x_4 : x_4^2] = [z_1^2 : z_2^2 : z_1 z_2 : z_3 z_4 : z_4^2] \right\}. \end{aligned}$$

The equality

$$[x_1^2 : x_2^2 : x_1 x_2 : x_3 x_4 : x_4^2] = [z_1^2 : z_2^2 : z_1 z_2 : z_3 z_4 : z_4^2]$$

in $\mathbb{P}_{\mathbb{C}}^4$ states two possibilities

$$[z_1 : z_2 : z_3 : z_4] = [x_1 : x_2 : x_3 : x_4] \text{ or } [z_1 : z_2 : z_3 : z_4] = [x_1 : x_2 : -x_3 : -x_4].$$

In this way, $X = A \cup B$, where

$$A = \left\{ ([x_1 : x_2 : x_3 : x_4], [x_1 : x_2 : x_3 : x_4]) \in (\mathbb{P}_{\mathbb{C}}^3)^2 : \right. \\ \left. x_1^2 + x_2^2 + x_3^2 = 0, \lambda x_1^2 + x_2^2 + x_4^2 = 0 \right\}$$

$$B = \left\{ ([x_1 : x_2 : x_3 : x_4], [x_1 : x_2 : -x_3 : -x_4]) \in (\mathbb{P}_{\mathbb{C}}^3)^2 : \right. \\ \left. x_1^2 + x_2^2 + x_3^2 = 0, \lambda x_1^2 + x_2^2 + x_4^2 = 0 \right\}$$

Clearly, $A \cap B = \emptyset$ and A and B are isomorphic to S .

The following example shows that, even in the case that the fiber product is connected, it might be reducible.

Example 2. If $S_1 = S_2 = S_0 = \widehat{\mathbb{C}}$ and $\beta_1(z) = \beta_2(z) = z(z^3 + z^2 + 1)$, then $S_1 \times_{(\beta_1, \beta_2)} S_2$ consists of two irreducible components, one of them is the Riemann sphere and the other is provided by the genus one Riemann surface

$$1 + x^2 + x^3 + xy + x^2y + y^2 + xy^2 + y^3 = 0.$$

Next result states sufficient conditions for the fiber product, when it is connected (for instance, if S_0 has genus zero), to be irreducible.

Theorem 2.6. *Let $\beta_1 : S_1 \rightarrow S_0$ and $\beta_2 : S_2 \rightarrow S_0$ be two non-constant holomorphic maps between closed Riemann surfaces so that the fiber product is connected. For each $j = 1, 2$ and each $q \in S_0$, we set*

$$a_q^{(j)} := \text{lcm}(\text{ord}_{\beta_j}(z) : \beta_j(z) = q)$$

where “lcm” stands for “least common multiple”. If either

- (1) $\gcd(\deg(\beta_1), \deg(\beta_2)) = 1$; or
- (2) $\gcd(a_q^{(1)}, a_q^{(2)}) = 1$, for every $q \in S_0$,

where “gcd” stands for “greater common divisor”, then the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ is irreducible, in particular, $S_1 \times_{(\beta_1, \beta_2)} S_2$ is a closed Riemann surface.

Proof. The singular points of $S_1 \times_{(\beta_1, \beta_2)} S_2$ are those points $(z_1, z_2) \in S_1 \times S_2$ such that $\beta_1(z_1) = \beta_2(z_2)$ and so that, for each $j = 1, 2$, the point $z_j \in S_j$ is a critical point of β_j , that is, $\text{ord}_{\beta_j}(z_j) = n_j \geq 2$. In local coordinates, we may assume that the singular point is $(0, 0)$ and that $\beta_j(z) = z^{n_j}$; so a neighborhood of such a singular point looks (locally) like $\{(z, w) \in \mathbb{C}^2 : z^{n_1} = w^{n_2}\}$. In this way, if $\gcd(n_1, n_2) = d$, then the singular point (z_1, z_2) has a neighborhood that looks like d different cones glued along such a point.

Let us assume (1) holds. Let R_k be any of the irreducible components of $S_1 \times_{(\beta_1, \beta_2)} S_2$ and set $d_{1,k} = \deg(\pi_1 : R_k \rightarrow S_1)$ and $d_{2,k} = \deg(\pi_2 : R_k \rightarrow S_2)$. Then, $d_{1,k} \cdot \deg(\beta_1) = d_{2,k} \cdot \deg(\beta_2)$. As $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2 : S_1 \times_{(\beta_1, \beta_2)} S_2 \rightarrow \widehat{\mathbb{C}}$ has degree $\deg(\beta_1)\deg(\beta_2)$, one has that $d_{1,k} \leq \deg(\beta_2)$ and $d_{2,k} \leq \deg(\beta_1)$. The condition $\gcd(\deg(\beta_1), \deg(\beta_2)) = 1$ asserts that $d_{j,k} = \deg(\beta_j)$ and, in particular, that there is at most one such irreducible component of $S_1 \times_{(\beta_1, \beta_2)} S_2$.

Let us now assume (2) holds. Under the hypothesis, each of the singular points of $S_1 \times_{(\beta_1, \beta_2)} S_2$ has a neighborhood homeomorphic to a disc. As already we know that $S_1 \times_{(\beta_1, \beta_2)} S_2$ is connected, the result follows. \square

Next example shows that the sufficient conditions in Theorem 2.6 are not necessary ones.

Example 3 (An example where the conditions of Theorem 2.6 are not necessary). Let us consider $S_1 = S_2 = \widehat{\mathbb{C}}$, $\beta_1(z) = 4z^3(1 - z^3)$ and $\beta_2(w) = -27w^4(w^2 - 1)/4$. In this case, $\deg(\beta_1) = 6 = \deg(\beta_2)$, so Condition (1) of Theorem 2.6 does not hold. Also, as $\beta_1^{-1}(\infty) = \{\infty\} = \beta_2^{-1}(\infty)$ and $\text{ord}_{\beta_1}(\infty) = 6 = \text{ord}_{\beta_2}(\infty)$, then neither Condition (2) of Theorem 2.6 holds. But,

$$S_1 \times_{(\beta_1, \beta_2)} S_2 = \{[z : w : t] \in \mathbb{P}_{\mathbb{C}}^2 : 16z^3(t^3 - z^3) - 27w^4(w^2 - t^2) = 0\},$$

is an irreducible curve. It has a singular point with a neighborhood being 6 cones glued at their vertices (corresponding to the preimage of ∞) and another singular point (one of the preimages of 0) with a neighborhood being a disc. This curve has genus 7.

Theorem 2.6 provides sufficient conditions for the fiber product to be irreducible, but these conditions are not necessary as seen from Example 3. In most of the cases (without assuming the conditions in such a theorem) an upper bound for the number of irreducible components is provided by the following (as a consequence of the Fuchsian description).

Lemma 2.7. *The number of irreducible components of the fiber product of the two pairs (S_1, β_1) and (S_2, β_2) is at most $\gcd(\deg(\beta_1), \deg(\beta_2))$.*

3. THE STRONG FIELD OF MODULI OF THE FIBER PRODUCT

3.1. The field of moduli of pairs. Let us consider a pair (R, η) , where R is a closed Riemann surface and $\eta : R \rightarrow \widehat{\mathbb{C}}$ is a non-constant meromorphic map. As a consequence of the Riemann-Roch's Theorem [24], we may assume that R is a non-singular complex projective algebraic curve and that η is a rational map. From now on, we assume all our pairs to be described algebraically, say that R is given as the common zeroes of the homogeneous polynomials P_1, \dots, P_n and that $\eta = Q_1/Q_2$, where Q_1 and Q_2 are homogeneous polynomials of the same degree.

If $\sigma \in \text{Gal}(\mathbb{C})$ (the group of field automorphisms of \mathbb{C}), then we denote by P_j^σ (respectively, Q_j^σ) the polynomial obtained from P_j (respectively, Q_j) by applying σ to its coefficients. The polynomials $P_1^\sigma, \dots, P_n^\sigma$ define a non-singular complex projective algebraic curve R^σ (homeomorphic to R) and $\eta^\sigma = Q_1^\sigma / Q_2^\sigma$ is a rational map on it. We say that (R^σ, η^σ) is isomorphic to (R, η) if there is an isomorphism $f_\sigma : R \rightarrow R^\sigma$ so that $\eta^\sigma \circ f_\sigma = \eta$. We denote this by the symbol $(R^\sigma, \eta^\sigma) \equiv (R, \eta)$. The *field of moduli* of the pair (R, η) is defined as the fixed field of the group

$$G = \left\{ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : (R^\sigma, \eta^\sigma) \equiv (R, \eta) \right\}.$$

It is a well known fact that this field is contained in any field of definition of (R, η) , but it might be that it is not a field of definition (see, for instance, [8, 12, 13, 16, 19, 27]). A consequence of Weil's descent theorem [30] (see also [15]), the field of moduli is a field of definition if R has no non-trivial automorphisms. A result due to Wolfart [31] asserts that if R is quasilatonic (i.e., when $R/\text{Aut}(R)$ has genus zero and exactly three cone points) then the field of moduli is also a field of definition.

3.2. Strong field of moduli of the fiber product. Let us consider two pairs (S_1, β_1) and (S_2, β_2) , where S_1 and S_2 are closed Riemann surfaces and $\beta_1 : S_1 \rightarrow \widehat{\mathbb{C}}$ and $\beta_2 : S_2 \rightarrow \widehat{\mathbb{C}}$ are non-constant meromorphic maps. We have the corresponding (connected) fiber product $(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta)$, where $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2$ and π_j is the corresponding projection onto the j factor.

If $\sigma \in \text{Gal}(\mathbb{C})$, then we have the new pairs $(S_1^\sigma, \beta_1^\sigma)$, $(S_2^\sigma, \beta_2^\sigma)$ and their corresponding fiber product $(S_1^\sigma \times_{(\beta_1^\sigma, \beta_2^\sigma)} S_2^\sigma, \beta^\sigma)$. We say that $(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta)$ is equivalent to $(S_1^\sigma \times_{(\beta_1^\sigma, \beta_2^\sigma)} S_2^\sigma, \beta^\sigma)$, denoted this by the symbol $(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta) \equiv^s (S_1^\sigma \times_{(\beta_1^\sigma, \beta_2^\sigma)} S_2^\sigma, \beta^\sigma)$, if there are isomorphisms

$$\begin{aligned} F : S_1 \times_{(\beta_1, \beta_2)} S_2 &\rightarrow S_1^\sigma \times_{(\beta_1^\sigma, \beta_2^\sigma)} S_2^\sigma \\ F_1 : S_1 &\rightarrow S_1^\sigma, \quad F_2 : S_2 \rightarrow S_2^\sigma \end{aligned}$$

so that $\beta^\sigma \circ F = \beta$ and $\pi_j \circ F = F_j \circ \pi_j$ for $j = 1, 2$.

The *strong field of moduli* of the fiber product $(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta)$ is defined as the fixed field of the group

$$G = \left\{ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : (S_1 \times_{(\beta_1, \beta_2)} S_2, \beta) \equiv^s (S_1^\sigma \times_{(\beta_1^\sigma, \beta_2^\sigma)} S_2^\sigma, \beta^\sigma) \right\}$$

Remark 3.1. If in our previous definition we forget the existence of the isomorphisms F_1 and F_2 , then we obtain the field of moduli of the pair $(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta)$. So, we may see that the field of moduli of the fiber product is a subfield of its strong field of moduli. The strong field of moduli takes care of all the information in the construction of the fiber product.

It can be seen from the definitions that the strong field of moduli always contains the fields of moduli of (S_1, β_1) , (S_2, β_2) and of $(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta)$. The following result states that this is the smallest fields containing the fields of moduli of the two starting pairs.

Theorem 3.2. *The strong field of moduli of $(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta)$ is the smallest field containing the fields of moduli of the two starting pairs.*

Proof. Let us consider the subgroup $G_j = \{\sigma \in \text{Gal}(\mathbb{C}) : (S_j^\sigma, \beta_j^\sigma) \equiv (S_j, \beta_j)\}$ and its fixed field \mathbb{K}_j (the field of moduli of the pair (S_j, β_j)), for $j = 1, 2$. Let \mathbb{K} be the smallest field containing \mathbb{K}_1 and \mathbb{K}_2 . As already observed in Remark 3.1, we only need to prove that the strong field of moduli is contained in \mathbb{K} .

If $\sigma \in G_1 \cap G_2$, then, for each $j = 1, 2$, there is an isomorphism $f_j : S_j \rightarrow S_j^\sigma$ so that $\beta_j^\sigma \circ f_j = \beta_j$. Let us consider the isomorphism $f : S_1 \times_{(\beta_1, \beta_2)} S_2 \rightarrow S_1^\sigma \times_{(\beta_1^\sigma, \beta_2^\sigma)} S_2^\sigma$ defined by $f(z_1, z_2) = (f_1(z_1), f_2(z_2))$. It is clear that $(S_1 \times_{(\beta_1, \beta_2)} S_2)^\sigma = S_1^\sigma \times_{(\beta_1^\sigma, \beta_2^\sigma)} S_2^\sigma$ and that, if $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2$, then $\beta = \beta^\sigma \circ f$. This shows that the strong field of moduli of the fiber product of the starting pairs is contained in \mathbb{K} . \square

As already stated in Remark 3.1, the strong field of moduli differs from the field of moduli. In the following example we see that the strong field of moduli is an extension of degree two of the field of moduli for a fiber product.

Example 4. Let (S_1, β_1) be some pair defined over its field of moduli $\mathbb{Q}(i)$. Let $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) = \langle \sigma \rangle$, where $\sigma(i) = -i$. Set $S_2 = S_1^\sigma$ and $\beta_2 = \beta_1^\sigma$. Consider the pair (S_2, β_2) . In this case, we have the isomorphism $f : S_1 \times_{(\beta_1, \beta_2)} S_2 \rightarrow S_1^\sigma \times_{(\beta_1^\sigma, \beta_2^\sigma)} S_2^\sigma$ defined by $f(z_1, z_2) = (z_2, z_1)$. Clearly,

$$\beta^\sigma \circ f(z_1, z_2) = \beta^\sigma(z_2, z_1) = (\beta_1 \circ \pi_1)^\sigma(z_2, z_1) = \beta_1^\sigma(z_2) = \beta_2(z_2) = \beta(z_1, z_2).$$

Since there is no possible isomorphism between S_1 and S_1^σ (otherwise the field of moduli will be \mathbb{Q} , a contradiction), we may see that the strong field of moduli of the fiber product is still $\mathbb{Q}(i)$, but that the field of moduli of the above fiber product is \mathbb{Q} .

Question 3.3. *Is the strong field of moduli of the fiber product always a subfield of degree at most two of the field of moduli?*

3.3. An application to dessins d'enfants. Belyi's theorem [1] asserts that a closed Riemann surface S can be defined by a curve defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers if and only if there is a non-constant holomorphic map $\beta : S \rightarrow \widehat{\mathbb{C}}$ whose critical values are contained in the set $\{\infty, 0, 1\}$; we say that S is a *Belyi curve*, that β is a *Belyi map* for S and that (S, β) is a *Belyi pair*. Two Belyi pairs, (S_1, β_1) and (S_2, β_2) , are called equivalent if there exists an isomorphism (holomorphic homeomorphism) $h : S_1 \rightarrow S_2$ so that $\beta_2 \circ h = \beta_1$. In

this setting, Belyi's theorem asserts that every Belyi pair (S, β) is equivalent to a Belyi pair (C, β_C) , where the algebraic curve C and the rational map β_C are both defined over $\overline{\mathbb{Q}}$.

Let us consider two Belyi pairs (S_1, β_1) and (S_2, β_2) , where we assume S_j is given as an algebraic curve over $\overline{\mathbb{Q}}$ and β_j is a rational map also defined over $\overline{\mathbb{Q}}$. Then its fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ is a connected, possibly reducible, algebraic curve defined over $\overline{\mathbb{Q}}$ and the map $\beta : S_1 \times_{(\beta_1, \beta_2)} S_2 \rightarrow \widehat{\mathbb{C}}$ defined by $\beta(z_1, z_2) = \beta_j(z_j)$ is also a rational map defined over $\overline{\mathbb{Q}}$. Each irreducible component turns out to be a Belyi curve (and the restriction of β to it a Belyi map). By Theorem 2.6 we have simple sufficient conditions for the fiber product to be irreducible; so for $(S_1 \times_{(\beta_1, \beta_2)} S_2, \beta)$ to be a Belyi pair. In the Fuchsian description given in Section 2.3, the group Γ_0 can be chosen to be $\Gamma(2) = \langle A(z) = z + 2, B(z) = z/(2z + 1) \rangle$.

If $G_j < \text{Aut}(S_j)$ is the deck group of β_j , the direct product $G_1 \times G_2$ acts naturally on the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ by the rule

$$(g_1, g_2)(z_1, z_2) = (g_1(z_1), g_2(z_2)).$$

In this case, $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2 : S_1 \times_{(\beta_1, \beta_2)} S_2 \rightarrow \widehat{\mathbb{C}}$ satisfies that

- (1) $\beta(g_1, g_2) = \beta$, for every $(g_1, g_2) \in G_1 \times G_2$;
- (2) $\beta(z, w) = \beta(x, y)$ if and only if $(z, w) = (g_1, g_2)(x, y)$ for some $(g_1, g_2) \in G_1 \times G_2$.

Among all Belyi pairs the most interesting ones are the regular or quasiplatonic ones; these are the ones for which the Belyi map is a regular branched cover. If both Belyi pairs are regular, then the fiber product is also regular, all its irreducible components are isomorphic and the corresponding deck group is the direct product of both deck groups.

As defined by Grothendieck in [10], a *dessin d'enfant* is a bipartite map on a closed orientable surface. If D is a dessin d'enfant on the surface X and $f : X \rightarrow Y$ is an orientation preserving homeomorphism, then f defines, in a natural way, a dessin d'enfant $f(D)$ on the surface Y . Two dessins d'enfants, say D_1 and D_2 , where D_j is defined over the closed orientable surface X_j , are said to be *equivalent* if there is an orientation preserving homeomorphism $f : X_1 \rightarrow X_2$ so that $f(D_1) = D_2$. There is an equivalence between the categories of dessins d'enfants and Belyi pairs. If (S, β) is a Belyi pair, then on the topological underlying surface of S we have the dessin d'enfant obtained as the pre-image of the interval $[0, 1]$ under β (the pre-images of 0 are painted in one color black and the pre-images of 1 with color white). Conversely, by the Uniformization Theorem, if D is a dessin d'enfant on a closed orientable surface X , then there is a unique Riemann surface structure S on X and a Belyi map $\beta : S \rightarrow \widehat{\mathbb{C}}$ so that D is homotopic to the dessin obtained from β (so equivalent to D). In this setting, we have a fiber product of dessins d'enfant and the results of this paper permits to construct more dessins d'enfant by this process. Dessins d'enfant provides a combinatorial object over which the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully. We hope that this fiber product construction may be of interest to obtain new examples of dessins d'enfants and possible new invariants for them.

Let D be a dessin d'enfants on a closed orientable surface of genus g ; we assume the vertices are colored in either black or white. Associated to D is its valence

$$\text{val}(D) = (a_1, \dots, a_\alpha; b_1, \dots, b_\beta; c_1, \dots, c_\gamma)$$

where $1 \leq a_1 \leq a_2 \leq \dots \leq a_\alpha$ are the degrees at the black vertices of D (it has exactly α black vertices), $1 \leq b_1 \leq b_2 \leq \dots \leq b_\beta$ are the degrees at the white vertices of D (it has exactly β white vertices) and $1 \leq c_1 \leq c_2 \leq \dots \leq c_\gamma$ are the degrees at the faces of D (it has exactly γ faces). Remember that the degree of a face is half the number of boundary edges of it. By Euler's characteristic formula, $2 - 2g = \alpha + \beta + \gamma - n$, where $n = a_1 + \dots + a_\alpha = b_1 + \dots + b_\beta = c_1 + \dots + c_\gamma$ is the number of edges of D .

Due to the equivalence of the categories of Belyi pairs and dessins d'enfants, the above permit to talk of the fiber product of two given dessins. But, as already noted, such a fiber product may not be again a dessin d'enfants. Theorem 2.6 may be stated, in terms of dessins d'enfants, as follows.

Theorem 3.4. *Let D_1 and D_2 be two dessin d'enfants and let*

$$\text{val}(D_1) = (a_1^{(1)}, \dots, a_{\alpha_1}^{(1)}; b_1^{(1)}, \dots, b_{\beta_1}^{(1)}; c_1^{(1)}, \dots, c_{\gamma_1}^{(1)})$$

$$\text{val}(D_2) = (a_1^{(2)}, \dots, a_{\alpha_2}^{(2)}; b_1^{(2)}, \dots, b_{\beta_2}^{(2)}; c_1^{(2)}, \dots, c_{\gamma_2}^{(2)})$$

Define, for each $j = 1, 2$,

$$A_j := \text{lcm}(a_1^{(j)}, \dots, a_{\alpha_j}^{(j)}), \quad B_j := \text{lcm}(b_1^{(j)}, \dots, b_{\beta_j}^{(j)}), \quad C_j := \text{lcm}(c_1^{(j)}, \dots, c_{\gamma_j}^{(j)})$$

$$n_j = a_1^{(j)} + \dots + a_{\alpha_j}^{(j)} = b_1^{(j)} + \dots + b_{\beta_j}^{(j)} = c_1^{(j)} + \dots + c_{\gamma_j}^{(j)}$$

If either

- (1) $\gcd(n_1, n_2) = 1$; or
- (2) $\gcd(A_1, A_2) = \gcd(B_1, B_2) = \gcd(C_1, C_2) = 1$,

then the fiber product of the two dessins D_1 and D_2 is again a dessin d'enfants.

4. ISOGENOUS DECOMPOSITION OF THE JACOBIAN VARIETY OF FIBER PRODUCTS

Let G be a finite group acting on a closed Riemann surface S of genus $g_S \geq 2$. It is classically known that this action induces an action of G on the Jacobian variety JS of S and this, in turn, gives rise to a G -equivariant isogeny decomposition of JS into abelian subvarieties (see, for instance, [4, 7]).

The decomposition of Jacobian varieties with group actions has been extensively studied in different settings, with applications to theta functions, to the theory of integrable systems and to the moduli spaces of principal bundles of curves, among others. The simplest case of such a decomposition is when G is a group of order two; this fact was already noticed in 1895 by Wirtinger [29] and used by Schottky and Jung in [26]. For other special groups see, for example, [3, 5, 18, 21, 23, 25].

In [6], Kani and Rosen studied relations among idempotents in the algebra of rational endomorphisms of an arbitrary abelian variety. By means of these relations, in the case of the Jacobian variety of a Riemann surface S with action of a group G , they succeeded in proving a decomposition theorem for JS in which, under some assumptions, each factor is isogenous to the Jacobian of a quotient S_H of S by the action of a subgroup H of G . Recently, in [22], a generalization of Kani-Rosen's result was proved. For the sake of explicitness and for later use, we exhibit (a particular case of) this generalization.

Theorem 4.1 ([22]). *Let H_1, H_2 be groups of automorphisms of a Riemann surface C . Then*

$$JC \times JC_{\langle H_1, H_2 \rangle} \sim JC_{H_1} \times JC_{H_2} \times P$$

for some abelian subvariety P of JC .

Let us consider two pairs (S_1, β_1) and (S_2, β_2) , where S_j is a closed Riemann surface and $\beta_j : S_j \rightarrow S_0$ is a non-constant regular holomorphic map, over a closed Riemann surface S_0 , so that the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ is connected and irreducible. As a consequence of Theorem 4.1, under some conditions, in the next we provide an isogenous decomposition of the Jacobian $J(S_1 \times_{(\beta_1, \beta_2)} S_2)$ in such a way that it contains, simultaneously as factors, the Jacobians of the starting Riemann surfaces.

Theorem 4.2. *Let (S_1, β_1) and (S_2, β_2) be two pairs, where S_j is a closed Riemann surface and $\beta_j : S_j \rightarrow S_0$ is a non-constant regular holomorphic map, over a closed Riemann surface S_0 , so that the fiber product $S_1 \times_{(\beta_1, \beta_2)} S_2$ is non-singular (it might be non-connected).*

If C is a component of $S_1 \times_{(\beta_1, \beta_2)} S_2$ then

$$JC \times JS_0 \sim JS_1 \times JS_2 \times P$$

for a suitable abelian subvariety P of JS . In particular, if the genus of S_0 is zero, then

$$J(S_1 \times_{(\beta_1, \beta_2)} S_2) \sim JS_1 \times JS_2 \times P.$$

Proof. We recall that as the holomorphic maps β_j are regular, by Corollary 2.5, the irreducible components of $S_1 \times_{(\beta_1, \beta_2)} S_2$ are pairwise isomorphic closed Riemann surfaces; hence, their Jacobians are isomorphic (as principally polarized abelian varieties). It follows that it is enough to consider one of its irreducible components; say $C \cong \mathbb{H}^2/\Gamma_{12}$ where $\Gamma_{12} = \Gamma_1 \cap \Gamma_2$ and $\mathbb{H}^2/\Gamma_j \cong S_j$ for $j = 0, 1, 2$. Let us denote by $G_j \leq \text{Aut}(S_j)$ the deck group associated to the holomorphic map β_j . Clearly, Γ_j is a normal subgroup of Γ_0 and $G_j \cong \Gamma_0/\Gamma_j$.

As mentioned in the previous section, the direct product $G_1 \times G_2$ acts naturally on C and is isomorphic to the deck group associated to the holomorphic map $\beta : C \rightarrow S_0$, where $\beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2$. In terms of Fuchsian groups, this fact can be easily understood because

$$\Gamma_0/\Gamma_{12} \cong \Gamma_0/\Gamma_1 \times \Gamma_0/\Gamma_2 \cong G_1 \times G_2.$$

Consequently, the deck group $H_j \leq \text{Aut}(C)$ associated to the projection $\pi_j : C \rightarrow S_j$ is

$$H_1 \cong \Gamma_1/\Gamma_{12} \cong (\Gamma_1 \cap \Gamma_0)/(\Gamma_1 \cap \Gamma_2) \cong \{id\} \times G_2 \cong G_2$$

$$H_2 \cong \Gamma_2/\Gamma_{12} \cong (\Gamma_0 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2) \cong G_1 \times \{id\} \cong G_1.$$

Now, as the group generated by H_1 and H_2 is isomorphic to $G_1 \times G_2$, the quotient $C_{\langle H_1, H_2 \rangle}$ is isomorphic to S_0 . Thus, Theorem 4.1 ensures the existence of an abelian subvariety P of $J\mathcal{C}$ such that

$$J\mathcal{C} \times JS_0 \sim JS_1 \times JS_2 \times P.$$

Finally, if we assume S_0 to have genus zero, then $C = S_1 \times_{(\beta_1, \beta_2)} S_2$ and the result follows. \square

5. EXAMPLES

Example 5 (An example where the necessary conditions of Theorem 2.6 hold).

$$S_1 = \{[x : y : z] \in \mathbb{P}_{\mathbb{C}}^2 : y^3 - x^2z + xz^2 = 0\} \quad (\text{genus one curve})$$

$$S_2 = \{[x_1 : x_2 : x_3] \in \mathbb{P}_{\mathbb{C}}^2 : x_1^2 + x_2^2 + x_3^2 = 0\} \cong \widehat{\mathbb{C}}$$

$$\beta_1([x : y : z]) = \frac{x}{z}, \quad \beta_2([x_1 : x_2 : x_3]) = -\left(\frac{x_2}{x_1}\right)^2.$$

Note that (S_1, β_1) is a regular Belyi pair provided by the group $\langle [x : y : z] \mapsto [x : e^{2\pi i/3}y : z] \rangle \cong \mathbb{Z}_3$ and (S_2, β_2) is a regular Belyi pair provided by the group $\langle [x_1 : x_2 : x_3] \mapsto [-x_1 : x_2 : x_3], [x_1 : x_2 : x_3] \mapsto [x_1 : -x_2 : x_3] \rangle \cong \mathbb{Z}_2^2$.

As $\deg(\beta_1) = 3$, $\deg(\beta_2) = 4$, $\gcd(\deg(\beta_1), \deg(\beta_2)) = 1$, Theorem 2.6 asserts that the fiber product is irreducible; in fact

$$S_1 \times_{(\beta_1, \beta_2)} S_2 \\ \parallel$$

$$\{([x : y : z], [x_1 : x_2 : x_3]) : x_1^2 + x_2^2 + x_3^2 = 0, y^3 - x^2z + xz^2 = 0, xx_1^2 = -zx_2^2\} \subset \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^2,$$

which is isomorphic to the following irreducible curve of genus 4:

$$R = \{[y : v : w : t] : y^3t - v^4 - v^2t^2 = 0, t^2 + v^2 + w^2 = 0\} \subset \mathbb{P}_{\mathbb{C}}^3.$$

The surface S has the following automorphisms

$$T([y : v : w : t]) = [e^{2\pi i/3}y : v : w : t],$$

$$A([y : v : w : t]) = [y : -v : w : t], \quad B([y : v : w : t]) = [y : v : -w : t],$$

so that $\langle T, A, B \rangle = \langle T \rangle \times \langle A, B \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_2^2$. The map

$$F : R \rightarrow \widehat{\mathbb{C}} : [y : v : w : t] \mapsto x = \frac{w}{iv - t}$$

provides a regular branched cover with deck group $\langle T \rangle$. The branch values of F are given by the points $\infty, 0, \pm i, \pm 1$. It follows that R can be also described by the cyclic 3-gonal curve

$$y^3 = x(x^4 - 1).$$

The group $\langle A, B \rangle$, under the map F , corresponds in this model to the group $\langle a(x, y) = (1/x, -y/x^2), b(x) = (-x, -y) \rangle$.

Example 6 (An example where the upper bound in Lemma 2.7 is sharp).

$$S_1 = \{[x : y : z] \in \mathbb{P}_{\mathbb{C}}^2 : y^2 - x^2 + xz = 0\} \cong \widehat{\mathbb{C}}$$

$$S_2 = \{[x_1 : x_2 : x_3] \in \mathbb{P}_{\mathbb{C}}^2 : x_1^2 + x_2^2 + x_3^2 = 0\} \cong \widehat{\mathbb{C}}$$

$$\beta_1([x : y : z]) = \frac{x}{z} \quad \beta_2([x_1 : x_2 : x_3]) = -\left(\frac{x_2}{x_1}\right)^2.$$

In this example, $\deg(\beta_1) = 2$, $\deg(\beta_2) = 4$, $\gcd(\deg(\beta_1), \deg(\beta_2)) = 2$, (S_1, β_1) is a regular Belyi pair provided by the group $\langle [x : y : z] \mapsto [x : -y : z] \rangle \cong \mathbb{Z}_2$ and (S_2, β_2) is a regular Belyi pair provided by the group $\langle [x_1 : x_2 : x_3] \mapsto [-x_1 : x_2 : x_3], [x_1 : x_2 : x_3] \mapsto [x_1 : -x_2 : x_3] \rangle \cong \mathbb{Z}_2^2$. The fiber product is given by

$$S_1 \times_{(\beta_1, \beta_2)} S_2$$

$$\parallel$$

$$\{([x : y : z], [x_1 : x_2 : x_3]) : x_1^2 + x_2^2 + x_3^2 = 0, y^2 - x^2 + xz = 0, xx_1^2 = -zx_2^2\} \subset \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^2.$$

The above fiber product is reducible and it consists of two irreducible components, isomorphic to the following ones (both are isomorphic under the isomorphism $L([u : v : w : t]) = [u : v : -w : t]$):

$$R_1 = \{[u : v : w : t] : uw^2 + ut^2 + iwt^2 = 0, v^2 + w^2 + t^2 = 0\} \cong \widehat{\mathbb{C}}$$

$$R_2 = \{[u : v : w : t] : uw^2 + ut^2 - iwt^2 = 0, v^2 + w^2 + t^2 = 0\} \cong \widehat{\mathbb{C}}$$

Example 7 (An example where the upper bound in Lemma 2.7 is not sharp). Let us consider $S_1 = S_2 = \widehat{\mathbb{C}}$, $\beta_1(z) = 4z^3(1 - z^3) = \beta_2(z)$. In this case

$$S_1 \times_{(\beta_1, \beta_2)} S_2 = \{[z : w : t] : z^3(t^3 - z^3) = w^3(t^3 - w^3) = 0\},$$

which consists of three irreducible components, these being:

$$R_1 = \{[z : w : t] : z = w\} \cong \widehat{\mathbb{C}}$$

$$R_2 = \{[z : w : t] : z^2 + zw + w^2 = 0\} \cong \widehat{\mathbb{C}}$$

$$R_3 = \{[z : w : t] : z^3 + w^3 = t^3\} \quad (\text{a genus one curve}).$$

But, in this example we may see that $\gcd(\deg(\beta_1), \deg(\beta_2)) = 6 \neq 3$.

Example 8 (Fiber product of cyclic gonal curves). Let $n, m \geq 2$ be integers and set

$$\begin{aligned} S_1 &= \{[x : y : z] : y^n = x(x - z)z^{n-2}\} \\ S_2 &= \{[u : w : v] : w^m = u(u - v)v^{m-2}\} \\ \beta_1([x : y : z]) &= x/z, \quad \beta_2([u : w : v]) = u/v \end{aligned}$$

In this case,

$$\begin{aligned} &S_1 \times_{(\beta_1, \beta_2)} S_2 \\ &\parallel \\ &\{([x : y : z], [u : w : v]) : y^n = x(x - z)z^{n-2}, w^m = u(u - v)v^{m-2}, xv = zu\} \subset \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^2. \end{aligned}$$

In affine coordinates ($z = v = 1$), the above fiber product can be seen as follows:

$$X_{n,m} := \{(x, y, u, w) \in \mathbb{C}^4 : y^n = x(x - 1), w^m = y^n\}$$

If $D_{n,m} = \gcd(\deg(\beta_1), \deg(\beta_2))$, then writing

$$n = aD_{n,m}, \quad m = bD_{n,m}$$

we obtain that

$$X_{n,m} := \left\{ ((x, y, u, w) \in \mathbb{C}^4 : y^n = x(x - 1), \prod_{j=0}^{D_{n,m}-1} (w^b - \rho_{D_{n,m}}^j y^a) = 0) \right\},$$

where $\rho_{D_{n,m}}$ is a $D_{n,m}$ -th primitive root of the unity. It follows that $X_{n,m}$ (and so $S_1 \times_{(\beta_1, \beta_2)} S_2$) contains exactly $D_{n,m}$ connected components, these being given by

$$X_{n,m,j} := \{(x, y, u, w) \in \mathbb{C}^4 : y^n = x(x - 1), w^b = \rho_{D_{n,m}}^j y^a\}.$$

All of these irreducible components are isomorphic by Corollary 2.5 (see also [11]). In the following table, the last column provides the genus of the corresponding irreducible components in the fiber product.

n	m	$D_{n,m}$	genus
6	4	2	5
6	9	3	8
12	18	6	17

Example 9 (Fiber product of gonal curves). Let $n, m \geq 2$ be integers, $a, b \in \{1, \dots, n\}$, $c, d \in \{1, \dots, m - 1\}$, polynomials $P(y)$ of degree at most $n - 1$, $Q(w)$ of degree at most $m - 1$, and set

$$\begin{aligned} S_1 &= \{[x : y : z] : y^n + P(y) = x^a(x - z)^b z^{n-a-b}\} \\ S_2 &= \{[u : w : v] : w^m + Q(w) = u^c(u - v)^d v^{m-c-d}\} \\ \beta_1([x : y : z]) &= x/z \\ \beta_2([u : w : v]) &= u/v \end{aligned}$$

Work out as for the previous example. For instance, for $n = 6$, $m = 4$, $a = 1$, $b = 4$, $c = d = 1$, $P(y) = 0$ and $Q(w) = 0$, we get that the fiber product is irreducible of genus 9.

Example 10 (Fiber product of Fermat curves). In this example we consider the Fermat curves

$$\begin{aligned} S_1 &= \{[x_1 : x_2 : x_3] \in \mathbb{P}_{\mathbb{C}}^2 : x_1^n + x_2^n + x_3^n = 0\} \\ S_2 &= \{[y_1 : y_2 : y_3] \in \mathbb{P}_{\mathbb{C}}^2 : y_1^m + y_2^m + y_3^m = 0\} \\ \beta_1([x_1 : x_2 : x_3]) &= -\left(\frac{x_2}{x_1}\right)^n \quad \beta_2([y_1 : y_2 : y_3]) = -\left(\frac{y_2}{y_1}\right)^m. \end{aligned}$$

The pair (S_1, β_1) (respectively, (S_2, β_2)) is a regular Belyi pair whose deck group is the Abelian group \mathbb{Z}_n^2 (respectively, \mathbb{Z}_m^2). The fiber product is given by

$$\begin{aligned} &S_1 \times_{(\beta_1, \beta_2)} S_2 \\ &\parallel \\ &\{([x_1 : x_2 : x_3], [y_1 : y_2 : y_3]) : x_1^n + x_2^n + x_3^n = 0, y_1^m + y_2^m + y_3^m = 0, x_2^n y_1^m = x_1^n y_2^m\} \subset \mathbb{P}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^2. \end{aligned}$$

(I) If we assume that $\gcd(n, m) = 1$, then we know that $S_1 \times_{(\beta_1, \beta_2)} S_2$ is irreducible and has exactly $3nm$ singular points. In fact, this represent the Belyi curve of genus $(2+n^2m^2-3nm)/2$ given by the Fermat curve

$$\{[z_1 : z_2 : z_3] \in \mathbb{P}_{\mathbb{C}}^2 : z_1^{nm} + z_2^{nm} + z_3^{nm} = 0\}$$

the Belyi map β is given by

$$\beta([z_1 : z_2 : z_3]) = -\left(\frac{z_2}{z_1}\right)^{nm}.$$

(II) If we assume that $\gcd(n, m) = d > 1$, then one has that one of the irreducible components of $S_1 \times_{(\beta_1, \beta_2)} S_2$ is given by the Fermat curve

$$\{[w_1 : w_2 : w_3] \in \mathbb{P}_{\mathbb{C}}^2 : w_1^{nm/d} + w_2^{nm/d} + w_3^{nm/d} = 0\}.$$

Remark 5.1. All other irreducible components are isomorphic to the previous one by Corollary 2.5.

For instance, if we set $n = 4$ and $m = 2$, then the fiber product provides 4 (isomorphic) irreducible components, each one a curve of genus 3, these being

$$\begin{aligned} R_1 &= \{[x : y : w : u : v] : x^2 + yw = 0, u^2 + vw = 0, y^2 + v^2 + w^2 = 0\} \\ R_2 &= \{[x : y : w : u : v] : x^2 + yw = 0, u^2 - vw = 0, y^2 + v^2 + w^2 = 0\} \\ R_3 &= \{[x : y : w : u : v] : x^2 - yw = 0, u^2 + vw = 0, y^2 + v^2 + w^2 = 0\} \end{aligned}$$

$$R_4 = \{[x : y : w : u : v] : x^2 - yw = 0, u^2 - vw = 0, y^2 + v^2 + w^2 = 0\}$$

An isomorphism between R_1 and R_2 is given by taking u to iu , an isomorphism between R_1 and R_3 is given by taking x to ix and an isomorphism between R_1 and R_4 is given by taking u to iu and x to ix . Moreover, R_1 is isomorphic to

$$\{[x : u : w] : x^4 + u^4 + w^4 = 0\}$$

by taking $y = -\frac{x^2}{w}$ and $v = -\frac{u^2}{w}$.

Example 11. In this example we consider

$$S_1 = \{[x : y : z] : y^7 = xz^4(x - z)^2\}$$

$$S_2 = \{[u : w : v] : w^3 = uv(u - v)\}$$

$$\beta_1([x : y : z]) = \frac{x}{z}, \quad \beta_2([u : w : v]) = \frac{u}{v}$$

The pair (S_1, β_1) (respectively, (S_2, β_2)) is a regular Belyi pair whose deck group is the Abelian group \mathbb{Z}_7 (respectively, \mathbb{Z}_3). The fiber product in this case has exactly 3 singular points; one over ∞ , other over 0 and the other over 1, and it is irreducible of genus 10. This provides then a regular Belyi pair with deck group \mathbb{Z}_{21} . An affine model of this fiber product is given by

$$\{(x, y, w) : y^7 = x(x - 1)^2, w^3 = x(x - 1)\}$$

and a projective model is

$$\{[x : y : w : t] : y^7 - x^3t^4 + 2x^2t^5 - xt^6 = 0, w^3 - x^2t + xt^2 = 0\}.$$

The singular 3 points are given by the points $[0 : 0 : 0 : 1]$, $[1 : 0 : 0 : 1]$, $[1 : 0 : 0 : 0]$.

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